Standard Normal Distribution

If X is normally distributed with mean μ and variance σ^2, generally written as $X \sim N(\mu, \sigma^2)$

Now If we define $z = \frac{X-\mu}{\sigma}$

then $E(z) = E\left(\frac{X-\mu}{\sigma}\right) = 0$

and $V(z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} V(X) = 1.$

The variable z defined so is called standard normal variate and its p.d.f. is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty \qquad (1)$$

We can easily obtain $\phi(z)$ by replacing $\mu \to 0$ and $\sigma^2 \to 1$ in p.d.f. of Normal distribution.

Obviously, the mean and variance of the standard normal variable are respectively zero and one, and is denoted by $z \sim N(0,1)$.

The distribution function of standard normal variate is given by $\Phi(z) = P(z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}z^2} dz$

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Now I shall write small letter z as z and Capital letter Z as Z. kindly see the difference.

$$\therefore \Phi(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}z^2} dz.$$

~~Area Probability~~

Area Property of a Normal Probability Curve

The probability of a normal variate lying between two values x_1 and x_2 is given by

$$P(x_1 \le X \le x_2) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2} dz \qquad \left[\text{let } z = \frac{x-\mu}{\sigma}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{z_2} e^{-\frac{1}{2}z^2} dz - \int_0^{z_1} e^{-\frac{1}{2}z^2} dz \right]$$

$$= P(z_2) - P(z_1)$$

where the definite integral $P(z) = \frac{1}{\sqrt{2\pi}} \int_0^{z} e^{-\frac{1}{2}z^2} dz$ is known as normal probability integral and gives the area under standard normal curve between the ordinates at $Z=0$ and $Z=z$. These areas have been tabulated for different values of z at intervals of 0.01

and are given at Table at the end of book.

In particular, $P(\mu-\sigma \le X \le \mu+\sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\, dx$

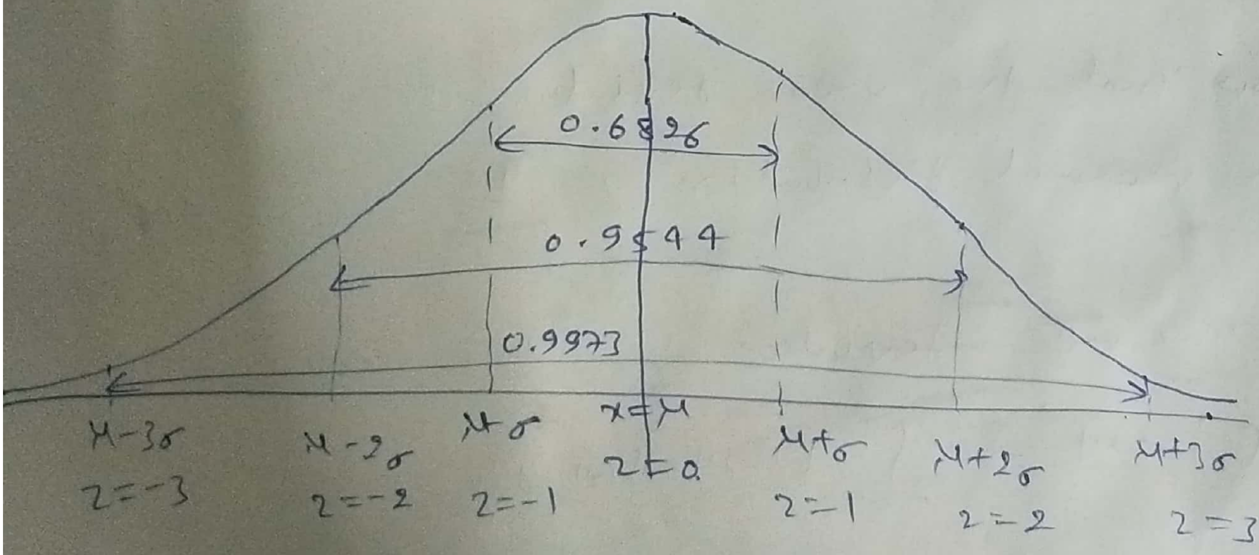
$\Rightarrow P(-1 \le z \le 1) = \int_{-1}^{1} \phi(z)\, dz = \frac{2}{\sqrt{2\pi}} \int_0^1 e^{-\frac{1}{2}z^2}\, dz$

$\Rightarrow P(-1 \le z \le 1) = 2 \times 0.3413 = 0.6826$ (from Table)

Similarity,

$P(\mu-2\sigma \le X \le \mu+2\sigma) = P(-2 \le z \le 2) = 0.9544$

and $P(\mu-3\sigma \le X \le \mu+3\sigma) = P(-3 \le z \le 3) = 0.9973$



Hence, the probability that a normal variate X lies outside the region $\mu \pm 3\sigma$ is given by

$P(|X-\mu| > 3\sigma) = P(|z| > 3) = 1 - P(-3 \le z \le 3)$

$= 1 - 0.9973 = 0.0027$ (very near to zero)

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Thus, we can easily see that theoretically normal variate ranges from $-\infty$ to ∞, yet in all probability we should expect it to lie within the range $\mu \pm 3\sigma$

Note:

(i) From the symmetry of the normal probability curve, we have

$$P(z \geq 2) = P(z \leq -2)$$

(ii) We observe that $P(-1.96 < z < 1.96) = 0.95$ and $P(-2.58 < z < 2.58) = 0.99$ and these are two important area values to remember.

Q. Show that the mean deviation from the mean of a normal distribution is $\frac{4\sigma}{5}$ approximately.

Soln

The mean deviation from the mean μ is

$$E|X-\mu| = \int_{-\infty}^{\infty} |x-\mu| \, f(x) \, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} |x-\mu| \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| \, e^{-\frac{1}{2}z^2} dz \qquad \left[\text{Let } z = \frac{x-\mu}{\sigma}\right]$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} z e^{-\frac{z^2}{2}} dz + \int_{0}^{\infty} z e^{-z^2/2} dz \right) \qquad \left\{ \because |z| = \begin{cases} z, & \text{if } z > 0 \\ -z, & \text{if } z < 0 \end{cases} \right.$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty z\, e^{-z^2/2} = \sigma\sqrt{\frac{2}{\pi}} \left[-e^{-\frac{1}{2}z^2}\right]_0^\infty$$

$$= \sigma\sqrt{\frac{2}{\pi}} = 0.7979\,\sigma \simeq \frac{4\sigma}{5}.$$

Moment generating function of standard Normal Dist.

$\because$ M.g.f. of Normal dist. is

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$\because$ for standard normal, $\mu = 0$ & $\sigma = 1$

$$\therefore M_Z(t) = e^{t^2/2}.$$

Q. Show that Normal distribution is a limiting case of Poisson distribution.

Soln

Let $X \sim Poi(\lambda)$,

then p.m.f. of X is defined as

$$P(X=x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0,1,2,3,\ldots$$

The moment generating function of X is

$$M_X(t) = E(e^{tX}) = e^{\lambda(e^t - 1)} \quad \text{—— (1)}$$

Now consider a standardized Poisson random variable

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

[Recall that, for Poisson distribution
mean $\mu = \lambda$
Variance $\sigma^2 = \lambda$
$\therefore \sigma = \sqrt{\lambda}$]

Now we shall show that Mgf of $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ is same as Mgf of standard normal variate as $\lambda \to \infty$.

Now

$$\lim_{\lambda \to \infty} M_Z(t) = \lim_{\lambda \to \infty} M_{\frac{X-\lambda}{\sqrt{\lambda}}}(t)$$

$$= \lim_{\lambda \to \infty} E\left[e^{t\left(\frac{X-\lambda}{\sqrt{\lambda}}\right)}\right]$$

$$= \lim_{\lambda \to \infty} e^{-\sqrt{\lambda}t} \, E\left[e^{\frac{tX}{\sqrt{\lambda}}}\right]$$

$$= \lim_{\lambda \to \infty} e^{-\sqrt{\lambda}t} \cdot e^{\lambda(e^{t/\sqrt{\lambda}} - 1)}$$ [from eq^n (1)]

$$= \lim_{\lambda \to \infty} \exp\left[-\sqrt{\lambda}t + \lambda\left\{\left(1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2!\,\lambda} + \frac{t^3}{3!\,\lambda^{3/2}} + \cdots\right) - 1\right\}\right]$$

$$= \lim_{\lambda \to \infty} \exp\left[\frac{t^2}{2} + \frac{t^3}{6}\lambda^{-1/2} + \cdots\right]$$

$$= \exp\left(t^2/2\right), \quad -\infty < Z < \infty$$

Same as the Mgf of standard normal distribution.

This implies that the associated unstandardized random variable X has a limiting distribution that is normal with mean λ and variance λ. Thus, we can say that ~~p~~ Normal distribution is a limiting case of Poisson Distribution.

Q. Show that Normal distribution is a limiting case of Binomial Distribution

Solⁿ Try to do yourself

Hints: use $Z = \frac{X - np}{\sqrt{npq}}$

$$M_Z(t) = E\left(e^{t\left(\frac{X-np}{\sqrt{npq}}\right)}\right) = e^{-\frac{npt}{\sqrt{npq}}}\left[q + p\,e^{\frac{t}{\sqrt{npq}}}\right]^n$$

$$\log M_Z(t) = -\frac{npt}{\sqrt{npq}} + n\log\left[q + p\,e^{\frac{t}{\sqrt{npq}}}\right]$$

try to prove.

$$\lim_{n\to\infty} \log M_Z(t) = t^2/2$$

$$\Rightarrow \lim_{n\to\infty} M_Z(t) = e^{t^2/2}$$, Same as the mgf of standard Normal Distribution